The Vasicek Interest Rate Process
Part VII - Zero Coupon Bond Price Derivatives

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January, 2021

In Part VII of the Vasicek Interest Rate Process series we will calculate the derivatives of the price of a pure discount bond that was purchased at time s and matures at time t. To this end we will work through the following hypothetical problem...

Our Hypothetical Problem

We are tasked with calculating the change in price of a zero coupon bond via a Taylor Series Expansion of the second order. The go-forward model assumptions are...

Description	Symbol	Value
Face value (in dollars)	F	1,000
Current short rate	r_s	0.04
Long-term short rate mean	r_∞	0.09
Annualized short rate volatility	σ	0.03
Mean reversion rate	λ	0.35

Question 1: Using the parameters above, what are the first and second order derivatives of bond price given that the bond was purchased at the end of year 3 and matures at the end of year 7?

Question 2: Using first and second order derivatives what is the change in bond price assuming that the change in time is one month and the change in the short rate is 10 bps?

Question 3: How accurate was your estimate of the change in price using bond derivatives?

Standard Zero Coupon Bond Price Equations

The short rate is the annualized interest rate at which an entity can borrow money for an infinitesimally short period of time. Vasicek's stochastic differential equation that defines the change in the short rate r_u in over the infinitesimally small time interval $[u, u + \delta u]$ is... [1]

$$\delta r_u = \lambda \, (r_\infty - r_u) \, \delta u + \sigma \, \delta W_u \tag{1}$$

The short rate at time t is defined as the short rate at time s (known) plus the sum of the changes in the short rate over the time period $[s, t]$ (random). Using Equation (1) above the equation for the random short rate at time t as a function of the short rate at time s is... [1]

$$r_t = r_s + \int\limits_s^t \delta r_u \;\; \text{...where...} \;\; t > s \tag{2}$$

The equation for the expected short rate at time t from the perspective of time zero is [1]

$$\mathbb{E}\Bigg[r_t\Bigg] = r_\infty + \text{Exp}\Bigg\{ -\lambda \, t \Bigg\} (r_0 - r_\infty) \tag{3}$$

We defined the variable $R_{s,t}$ to be the stochastic discount rate over the time interval $[s,t]$. The equation for the stochastic discount rate is... [2]

$$R_{s,t} = \int\limits_{s}^{t} r_u \, \delta u \tag{4}$$

We defined the variable m to be the mean of the stochastic discount rate $R_{s,t}$. The equation for the mean of the stochastic discount rate is... [2]

$$m = r_\infty \, (t-s) + (r_\infty - r_s) \left(\mathrm{Exp} \left\{ -\lambda \, (t-s) \right\} - 1 \right) \lambda^{-1} \tag{5}$$

We defined the variable v to be the variance of the stochastic discount rate $R_{s,t}$. The equation for the variance of the stochastic discount rate is... [2]

$$v = \frac{\sigma^2}{2\,\lambda^3} \left(2\,\lambda \, (t-s) - 3 + 4 \, \mathrm{Exp} \left\{ -\lambda \, (t-s) \right\} - \mathrm{Exp} \left\{ -2\,\lambda \, (t-s) \right\} \right) \tag{6}$$

In Equations (5) and (6) above r_s is the short rate at time s, r_∞ is the long-term short rate, λ is the rate of mean reversion, and σ is the annual short rate volatility.

We will define the variable $P(s,t)$ to be the price of a \$1.00 face value zero coupon bond purchased at time s and matures at time t. Using the equations above the equation for zero coupon bond price at time s is... [3]

$$P(s,t) = \mathbb{E}_s^Q \left[\mathrm{Exp} \left\{ -\int\limits_{s}^{t} r_u \, \delta u \right\} \right] = \mathrm{Exp} \left\{ -R_{s,t} \right\} = \mathrm{Exp} \left\{ -m + \frac{1}{2} \, v \right\} \tag{7}$$

Alternative Zero Coupon Bond Price Equations

We will define the following bond price variables... [4]

$$A(s,t) = \left(r_\infty - \frac{\sigma^2}{2\,\lambda^2} \right) \left(B(s,t) - (t-s) \right) - \frac{\sigma^2}{4\,\lambda} \, B^2(s,t) \;\; \text{...and...} \;\; B(s,t) = \left(1 - \mathrm{Exp} \left\{ -\lambda \, (t-s) \right\} \right) \lambda^{-1} \tag{8}$$

Using Equations (5), (6) and (8) above we can make the following statement... [4]

$$A(s,t) - B(s,t) \, r_s = -m + \frac{1}{2} \, v \tag{9}$$

Using Equation (9) above we can rewrite zero coupon bond price Equation (7) above as... [4]

$$P(s,t) = \mathrm{Exp} \left\{ -m + \frac{1}{2} \, v \right\} = \mathrm{Exp} \left\{ \theta_s \right\} \;\; \text{...where...} \;\; \theta_s = A(s,t) - B(s,t) \, r_s \tag{10}$$

Bond Price Derivatives

The equation for the change in zero coupon bond price over the infinitesimally small time interval $[s, s+\delta s]$ is...

$$\delta P(s,t) = \frac{\delta P(s,t)}{\delta \theta_s} \, \frac{\delta \theta_s}{\delta s} \, \delta s + \frac{\delta P(s,t)}{\delta \theta_s} \, \frac{\delta \theta_s}{\delta r_s} \, \delta r_s = P(s,t) \left(\frac{\delta \theta_s}{\delta s} \, \delta s + \frac{\delta \theta_s}{\delta r_s} \, \delta r_s \right) \tag{11}$$

The equations for the relevant derivatives of Equation (10) above are...

$$\frac{\delta \theta_s}{\delta s} = \frac{\delta A(s,t)}{\delta s} - \frac{B(s,t) \, r_s}{\delta s} \;\; \text{...and...} \;\; \frac{\delta \theta_s}{\delta r_s} = \frac{\delta A(s,t)}{\delta r_s} - \frac{B(s,t) \, r_s}{\delta r_s} \tag{12}$$

Note the following derivatives using Appendix Equations (23), (24), (33) and (34) below...

$$f'(s) = \frac{\delta}{\delta s}\,A(s,t) = \left(r_\infty - \frac{\sigma^2}{2\,\lambda^2}\right)\left(1 - \mathrm{Exp}\left\{-\lambda\,(t-s)\right\}\right) - \frac{1}{2}\,\sigma^2\left(\mathrm{Exp}\left\{-2\,\lambda\,(t-s)\right\} - \mathrm{Exp}\left\{-\lambda\,(t-s)\right\}\right)$$
$$f''(s) = \frac{\delta^2}{\delta s^2}\,A(s,t) = -\left(r_\infty - \frac{\sigma^2}{2\,\lambda^2}\right)\lambda\,\mathrm{Exp}\left\{-\lambda\,(t-s)\right\} - \frac{1}{2}\,\sigma^2\left(2\,\lambda\,\mathrm{Exp}\left\{-2\,\lambda\,(t-s)\right\} - \lambda\,\mathrm{Exp}\left\{-\lambda\,(t-s)\right\}\right)$$
$$g'(s) = \frac{\delta}{\delta s}\,B(s,t)\,r_s = -\mathrm{Exp}\left\{-\lambda\,(t-s)\right\}r_s$$
$$g''(s) = \frac{\delta^2}{\delta s^2}\,B(s,t)\,r_s = -\lambda\,\mathrm{Exp}\left\{-\lambda\,(t-s)\right\}r_s \tag{13}$$

Note the following derivatives using Appendix Equations (20), (21), (27) and (28) below...

$$h'(r_s) = \frac{\delta}{\delta r_s}\,A(s,t) = 0$$
$$h''(r_s) = \frac{\delta^2}{\delta r_s^2}\,A(s,t) = 0$$
$$i'(r_s) = \frac{\delta}{\delta r_s}\,B(s,t)\,r_s = \left(1 - \mathrm{Exp}\left\{-\lambda\,(t-s)\right\}\right)\lambda^{-1}$$
$$i''(r_s) = \frac{\delta^2}{\delta r_s^2}\,B(s,t)\,r_s = 0 \tag{14}$$

Using Equations (13) and (14) above the equation for the change in bond price is...

$$\delta P(s,t) = P(s,t)\left[\left(f'(s) - g'(s)\right)\delta s + \frac{1}{2}\left(f''(s) - g''(s)\right)\delta s^2 - i'(r_s)\,\delta r_s\right] \tag{15}$$

Answers To Our Hypothetical Problem

Using expected short rate Equation (3) and the model parameters above the expected short rate at the beginning of the time interval $[3, 7]$ is...

$$\mathbb{E}\left[r_3\right] = 0.09 + \mathrm{Exp}\left\{-0.35 \times 3\right\} \times (0.04 - 0.09) = 0.0725 \tag{16}$$

Using the parameters to our hypothetical problem the equations for the change in time (δs) and change in rate (δr_s) are...

$$\delta s = \text{one month} = 0.08333 \text{ years } \ \text{...and...} \ \delta r_s = 10 \text{ bps} = 0.00100 \tag{17}$$

Hypothetical problem parameters:

Parameter	Before	After	Change	Reference
Face value (F)	1,000.00	1,000.00		
Bond purchase time (s)	3.00000	3.08333	0.08333	Equation (17)
Bond maturity time (t)	7.00000	7.00000		
Short rate	0.07250	0.07350	0.00100	Equations (16) and (17)
Long rate	0.09000	0.09000		
Mean reversion rate (lambda)	0.35000	0.35000		
Short rate volatility (sigma)	0.03000	0.03000		
Calculations:				
A(s,t)	-0.16246	-0.15701	0.00545	Equation (8)
B(s,t)	2.15258	2.13173	-0.02085	Equation (8)
Theta	-0.31853	-0.31370	0.00483	Equation (10)
Bond price ($)	727.22	730.74	3.52	Equation (10)

Question 1: What are the first and second order derivatives of bond price?

Using Equations (13), (14), (16) and (17) above the first and second order derivatives for bond price are...

Symbol	Equation	Derivative	Change	Product
$f'(s)$	$\frac{\delta}{\delta s}\,A(s,t)$	0.065122	0.08333	0.005427
$f''(s)$	$\frac{\delta^2}{\delta s^2}\,A(s,t)$	-0.022744	0.00694	-0.000158
$g'(s)$	$\frac{\delta}{\delta s}\,B(s,t)\,r_s$	-0.017879	0.08333	-0.001490
$g''(s)$	$\frac{\delta^2}{\delta s^2}\,B(s,t)\,r_s$	-0.006258	0.00694	-0.000043
$i'(r_s)$	$\frac{\delta}{\delta r_s}\,B(s,t)\,r_s$	2.152580	0.00100	0.002153

Question 2: Using first and second order derivatives what is the change in bond price assuming that the change in time is one month and the change in the short rate is 10 bps?

Using Equation (15) above the equation for the change in bond price is...

$$\delta P(s,t) = 727.22 \times \Bigg[\bigg(0.065122 - (-0.017879)\bigg) \times 0.08333 + \frac{1}{2} \times \bigg(-0.022744 - (-0.006258)\bigg) \times 0.08333^2 - 2.152580 \times 0.00100\Bigg] = 3.42277 \tag{18}$$

Question 3: How accurate was your estimate of the change in price using bond derivatives?

$$\begin{aligned} \text{Actual change in price} &= 3.52 \\ \text{Change in price via derivatives} &= 3.42 \\ \text{Dollar difference} &= 0.10 \end{aligned} \tag{19}$$

Note: There is a difference because the derivatives calculated above go out to the second order but the derivatives of bond price with respect to time are of an order greater than two.

Appendix

A. The equation for the first derivative of $B(s,t)$ in Equation (8) above with respect to the short rate at time s is...

$$\frac{\delta}{\delta r_s}\,B(s,t)\,r_s = \bigg(1 - \text{Exp}\bigg\{-\lambda\,(t-s)\bigg\}\bigg)\lambda^{-1} \tag{20}$$

Using Equation (20) above the second derivative of $B(s,t)$ with respect to the short rate at time s is...

$$\frac{\delta^2}{\delta r_s^2}\,B(s,t)\,r_s = 0 \tag{21}$$

B. The equation for the derivative of $B(s,t)$ in Equation (8) above with respect to time s is...

$$\begin{aligned} \frac{\delta}{\delta s}\,B(s,t) &= \frac{\delta}{\delta s}\Bigg[\bigg(1 - \text{Exp}\bigg\{-\lambda\,(t-s)\bigg\}\bigg)\lambda^{-1}\Bigg] \\ &= \frac{\delta}{\delta s}\Bigg[\lambda^{-1} - \lambda^{-1}\,\text{Exp}\bigg\{-\lambda\,t\bigg\}\,\text{Exp}\bigg\{\lambda\,s\bigg\}\Bigg] \\ &= \frac{\delta}{\delta s}\Bigg[\lambda^{-1}\Bigg] - \lambda^{-1}\,\text{Exp}\bigg\{-\lambda\,t\bigg\}\,\frac{\delta}{\delta s}\,\text{Exp}\bigg\{\lambda\,s\bigg\}\Bigg] \\ &= 0 - \text{Exp}\bigg\{-\lambda\,t\bigg\}\,\text{Exp}\bigg\{\lambda\,s\bigg\} \\ &= -\text{Exp}\bigg\{-\lambda\,(t-s)\bigg\} \end{aligned} \tag{22}$$

Using Equation (22) above the equation for the derivative of $B(s,t)\,r_s$ with respect to time s is...

$$\frac{\delta}{\delta s}\,B(s,t)\,r_s = -\text{Exp}\bigg\{-\lambda\,(t-s)\bigg\}\,r_s \tag{23}$$

Using Equation (23) above the second derivative of $B(s,t)$ with respect to time is...

$$\frac{\delta^2}{\delta s^2}\,B(s,t)\,r_s = -\lambda\,\mathrm{Exp}\left\{-\lambda\,(t-s)\right\}r_s \tag{24}$$

C. The equation for the square of $B(s,t)$ in Equation (8) above is...

$$\begin{aligned}
B^2(s,t) &= \left(\left(1-\mathrm{Exp}\left\{-\lambda\,(t-s)\right\}\right)\lambda^{-1}\right]^2 \\
&= \left(\frac{1}{\lambda}-\frac{1}{\lambda}\,\mathrm{Exp}\left\{-\lambda\,(t-s)\right\}\right)^2 \\
&= \frac{1}{\lambda^2}-\frac{2}{\lambda^2}\,\mathrm{Exp}\left\{-\lambda\,(t-s)\right\}+\frac{1}{\lambda^2}\,\mathrm{Exp}\left\{-2\,\lambda\,(t-s)\right\} \\
&= \left(1-2\,\mathrm{Exp}\left\{-\lambda\,(t-s)\right\}+\mathrm{Exp}\left\{-2\,\lambda\,(t-s)\right\}\right)\lambda^{-2} \\
&= \left(1-2\,\mathrm{Exp}\left\{-\lambda\,t\right\}\mathrm{Exp}\left\{\lambda\,s\right\}+\mathrm{Exp}\left\{-2\,\lambda\,t\right\}\mathrm{Exp}\left\{2\,\lambda\,s\right\}\right)\lambda^{-2}
\end{aligned} \tag{25}$$

The equation for the derivative of Equation (25) above with respect to time s is...

$$\begin{aligned}
\frac{\delta}{\delta s}\,B^2(s,t) &= \left(0-2\,\lambda\,\mathrm{Exp}\left\{-\lambda\,t\right\}\mathrm{Exp}\left\{\lambda\,s\right\}+2\,\lambda\,\mathrm{Exp}\left\{-2\,\lambda\,t\right\}\mathrm{Exp}\left\{2\,\lambda\,s\right\}\right)\lambda^{-2} \\
&= -2\,\lambda\left(\mathrm{Exp}\left\{-\lambda\,(t-s)\right\}-\mathrm{Exp}\left\{-2\,\lambda\,(t-s)\right\}\right)\lambda^{-2} \\
&= 2\left(\mathrm{Exp}\left\{-2\,\lambda\,(t-s)\right\}-\mathrm{Exp}\left\{-\lambda\,(t-s)\right\}\right)\lambda^{-1}
\end{aligned} \tag{26}$$

D. The equation for the first derivative of $A(s,t)$ in Equation (8) above with respect to the short rate at time s is...

$$\frac{\delta}{\delta r_s}\,A(s,t) = 0 \tag{27}$$

Using Equation (27) above the second derivative of $A(s,t)$ with respect to the short rate at time s is...

$$\frac{\delta^2}{\delta r_s^2}\,A(s,t) = 0 \tag{28}$$

E. The equation for the derivative of $A(s,t)$ in Equation (8) above with respect to time s is...

$$\begin{aligned}
\frac{\delta}{\delta s}\,A(s,t) &= \frac{\delta}{\delta s}\left(r_\infty-\frac{\sigma^2}{2\,\lambda^2}\right)B(s,t)-\frac{\delta}{\delta s}\left(r_\infty-\frac{\sigma^2}{2\,\lambda^2}\right)t+\frac{\delta}{\delta s}\left(r_\infty-\frac{\sigma^2}{2\,\lambda^2}\right)s-\frac{\delta}{\delta s}\,\frac{\sigma^2}{4\,\lambda}\,B^2(s,t) \\
&= \left(r_\infty-\frac{\sigma^2}{2\,\lambda^2}\right)\frac{\delta}{\delta s}\,B(s,t)+\left(r_\infty-\frac{\sigma^2}{2\,\lambda^2}\right)\frac{\delta}{\delta s}\,s-\frac{\sigma^2}{4\,\lambda}\,\frac{\delta}{\delta s}\,B^2(s,t)
\end{aligned} \tag{29}$$

Using Equation (23) above the solution to the first derivative in Equation (29) above is...

$$\left(r_\infty-\frac{\sigma^2}{2\,\lambda^2}\right)\frac{\delta}{\delta s}\,B(s,t) = -\left(r_\infty-\frac{\sigma^2}{2\,\lambda^2}\right)\mathrm{Exp}\left\{-\lambda\,(t-s)\right\} \tag{30}$$

The solution to the second derivative in Equation (29) above is...

$$\left(r_\infty-\frac{\sigma^2}{2\,\lambda^2}\right)\frac{\delta}{\delta s}\,s = r_\infty-\frac{\sigma^2}{2\,\lambda^2} \tag{31}$$

Using Equation (23) above the solution to the third derivative in Equation (29) above is...

$$\begin{aligned}
\frac{\sigma^2}{4\,\lambda}\,\frac{\delta}{\delta s}\,B^2(s,t) &= \frac{\sigma^2}{2\,\lambda}\left(\mathrm{Exp}\left\{-2\,\lambda\,(t-s)\right\}-\mathrm{Exp}\left\{-\lambda\,(t-s)\right\}\right)\lambda^{-1} \\
&= \frac{1}{2}\,\sigma^2\left(\mathrm{Exp}\left\{-2\,\lambda\,(t-s)\right\}-\mathrm{Exp}\left\{-\lambda\,(t-s)\right\}\right)
\end{aligned} \tag{32}$$

Using Equations (30), (31) and (32) above the solution to Equation (29) above is...

$$\frac{\delta}{\delta s}\,A(s,t)=\left(r_{\infty}-\frac{\sigma^2}{2\,\lambda^2}\right)\left(1-\mathrm{Exp}\left\{-\lambda\,(t-s)\right\}\right)-\frac{1}{2}\,\sigma^2\left(\mathrm{Exp}\left\{-2\,\lambda\,(t-s)\right\}-\mathrm{Exp}\left\{-\lambda\,(t-s)\right\}\right) \tag{33}$$

Using Equation (33) above the second derivative of $A(s,t)$ with respect to time is...

$$\frac{\delta^2}{\delta s^2}\,A(s,t)=-\left(r_{\infty}-\frac{\sigma^2}{2\,\lambda^2}\right)\lambda\,\mathrm{Exp}\left\{-\lambda\,(t-s)\right\}-\frac{1}{2}\,\sigma^2\left(2\,\lambda\,\mathrm{Exp}\left\{-2\,\lambda\,(t-s)\right\}-\lambda\,\mathrm{Exp}\left\{-\lambda\,(t-s)\right\}\right) \tag{34}$$

References

[1] Gary Schurman, *The Vasicek Interest Rate Process - The Stochastic Short Rate,* February, 2013.

[2] Gary Schurman, *The Vasicek Interest Rate Process - The Stochastic Discount Rate,* February, 2013.

[3] Gary Schurman, *The Vasicek Interest Rate Process - Zero Coupon Bond Price,* February, 2013.

[4] Gary Schurman, *The Vasicek Interest Rate Process - Alternative Zero Coupon Bond Price Equations,* February, 2013.